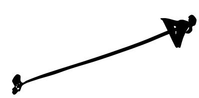
It's useful to familiarize with the notion of vectors.

Vectors are slightly more abstract version of straight arrows.

Here, by "straight arrows", we mean an arrow with a starting point and an endpoint:

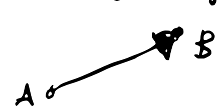


The concept of vectors came from physics, from the obvious realization that the physical law of the world stays the same regardless of where you stand.

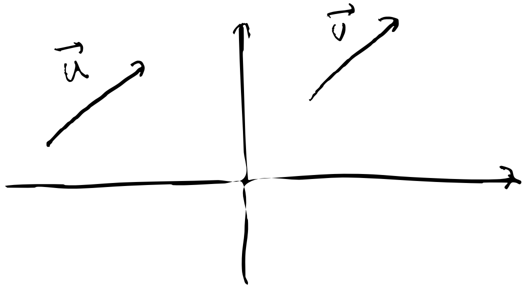
So, for example, physicists will not care about the position of the car, but about how fast it is moving (speed) and which direction it is moving to (direction).

This liberation from the information of "position" yields the concept of vector.

So vectors encode the data of magnitude and direction just like the case of cars moving. An arrow starting from point A ending at point B yields a vector called $\overrightarrow{AB}$



But vectors don't care about the position, so



two vectors $\vec{u}, \vec{v}$ can be equal even if they are situated at different places.

as long as <u>their magnitudes and their directions match.</u>

Vectors turn out to be extremely useful when dealing with many variables, so in particular in multivariable calculus!

<u>Examples</u>

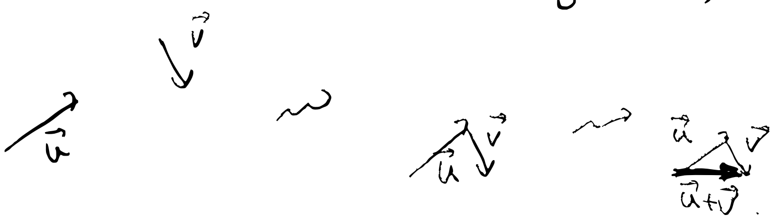
- Velocity (e.g. the car is moving 30mph north)
- Force (e.g. the gravity is applying the force of 100 Newtons downward on it.)

The real numbers are called **scalars**, which are just magnitudes, no direction. (30mph north : vector, 30 : scalar)
velocity / speed

In general, given a vector $\vec{v}$, its **length** (= magnitude) is denoted by $|\vec{v}|$, which is a scalar (no direction).

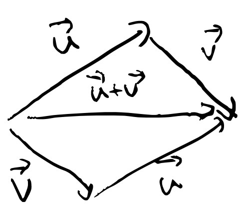
One can do various things with vectors.

<u>Adding vectors</u> The idea is simple. If you have $\vec{u}$ and $\vec{v}$, two vectors, $\vec{u}+\vec{v}$ is the vector from the starting point of $\vec{u}$ to the endpoint of $\vec{v}$, after you put them together so that $\vec{v}$ starts where $\vec{u}$ ends. (Triangle rule)



Namely, you follow two arrows concatenated, and where you'll end up is represented by the sum of two vectors.

<u>Note</u> The order doesn't matter just like numbers $(2+3=3+2)$ because



$\vec{u}+\vec{v}$ is the diagonal in a parallelogram.

(Parallelogram rule).

<u>Caution</u> There is no way that you can add a scalar and a vector! $2+\langle 3,1\rangle$, $|\vec{v}|+\vec{v}$, etc. all don't make sense.

Zero vector, negative of a vector

There's a special vector, called the zero vector, denoted $\vec{0}$.

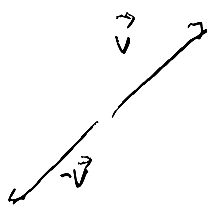
It's represented by an empty arrow, namely start point = endpoint.

No change, length is just zero.

For a vector $\vec{v}$, the negative of $\vec{v}$, denoted $-\vec{v}$,

is the vector obtained by taking the opposite direction with the

same magnitude.



__Exercise__ Explain why the following holds.

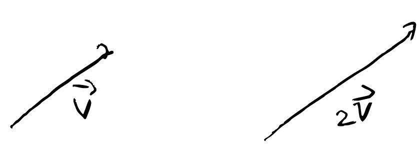
- $\vec{v} + \vec{0} = \vec{v}$
- $\vec{0} + \vec{v} = \vec{v}$
- $\vec{v} + (-\vec{v}) = \vec{0}$
- $(-\vec{v}) + \vec{v} = \vec{0}$

Scalar multiplication

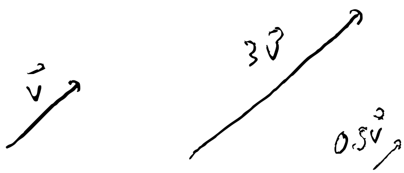
You can double a vector, or multiply a vector by any scalar (number like 2, 3.5, -3.14 ..).

For a vector $\vec{v}$, $2\vec{v}$ is the vector in the same

direction as $\vec{v}$, and is twice as long.

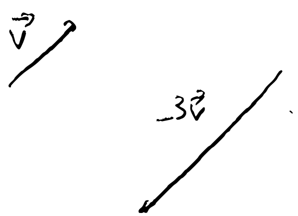


For any number c (scalar), $c\vec{v}$ is a vector in the direction as $\vec{v}$ and is c times as long.



EXCEPT what if $c=-3$? What is "-3 times as long"?

If c is negative, you flip the direction, and is $|c|$ times as long! Here, $|c|$ is the absolute value of c (remove any sign, so $|-3.22| = 3.22$, $|321.09| = 321.09$.)



What if $c=0$? Magnitude is 0, so it is $\vec{0}$.
$(0\vec{v} = \vec{0}.)$

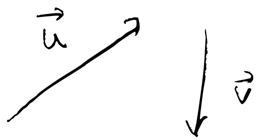
What if $c=-1$? This, flipping the direction and keeping the magnitude, is precisely the negative of $\vec{v}$. $(-1 \cdot \vec{v} = -\vec{v})$

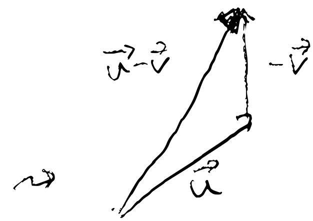
Definition

We say $\vec{u}, \vec{v}$ are parallel if one of them is a scalar multiple of the other. So $\vec{u} = c\vec{v}$ or $\vec{v} = d\vec{u}$.

Now the vectors seem to follow almost the same rules as numbers. For example you can subtract vectors:

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}).$$





Some basic rules ($\vec{u}, \vec{v}, \vec{w}$ are vectors. a, b are scalars)

- $a(b\vec{u}) = (ab)\cdot\vec{u}$ (like, $2(2\vec{u}) = 4\vec{u}$)
- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (sum can be switched)
- $\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$, $0\,\vec{v} = \vec{0}$, $1\vec{v} = \vec{v}$, $(-1)\,\vec{v} = -\vec{v}$.
- $\vec{v} + (-\vec{v}) = \vec{0}$.
- $a\vec{u} + b\vec{u} = (a+b)\vec{u}$, $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ (distribution rule)
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (sum can be done in any order)

Question Can you multiply two vectors?

Answer Yes and no. Unlike numbers, there are two ways to multiply (dot product and cross product), which we will learn shortly.

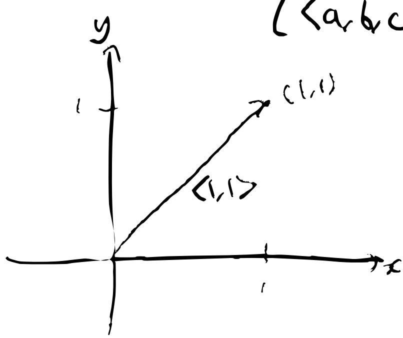
Expressing a vector using rectangular coordinates

Vectors are designed to work well with rectangular coordinates. You can express a vector in numbers using rectangular coordinates.

Remember that the position of a vector can be placed anywhere. So, given a vector $\vec{u}$, you can place the starting point to be the origin. Then the endpoint encodes the data of $\vec{u}$. If the endpoint is the point $\begin{cases} (a,b) & (\text{in } 2D) \\ (a,b,c) & (\text{in } 3D) \end{cases}$, then the vector $\vec{u}$ is expressed as $\begin{cases} \langle a,b \rangle & (\text{in } 2D) \\ \langle a,b,c \rangle & (\text{in } 3D) \end{cases}$.

These a,b,c are called **components** of the vector.

$$\boxed{\begin{array}{l} a: x\text{-component} \\ b: y\text{-component} \\ c: z\text{-component} \end{array}}$$



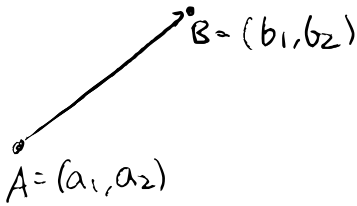
Here, note the difference in symbols we are using. Points use (,), while vectors use < , >.

So (1,1) (a point) and <1,1> (a vector) are different; in the end vectors don't care about where it's placed. So the arrow from point (1,9) to point (2,10) is also the same vector <1,1>.

Using the components, the vector arithmetic we discussed before becomes easy.

- The vector formed by the arrow

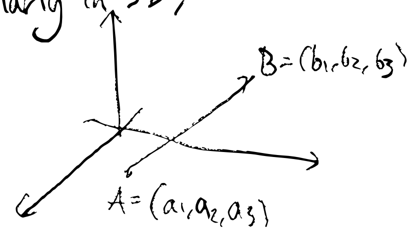
$\overrightarrow{AB}$



is $\langle b_1-a_1, b_2-a_2\rangle$

Similarly in 3D,

$\overrightarrow{AB}$



is $\langle b_1-a_1, b_2-a_2, b_3-a_3\rangle$.

- Adding, subtracting vectors, and multiplying vectors by scalar are all easily expressed: just do the arithmetic on each component.

$$\langle a_1,a_2,a_3\rangle+\langle b_1,b_2,b_3\rangle = \langle a_1+b_1, a_2+b_2, a_3+b_3\rangle$$
$$\langle a_1,a_2,a_3\rangle-\langle b_1,b_2,b_3\rangle = \langle a_1-b_1, a_2-b_2, a_3-b_3\rangle$$
$$c\langle a_1,a_2,a_3\rangle = \langle ca_1, ca_2, ca_3\rangle$$

<u>Example</u> The vector represented by the arrow starting from

$A=(2,-3,4)$ ending at $B=(-2,1,1)$ is

$\langle -2-2, 1-(-3), 1-4\rangle = \langle -4, 4, -3\rangle$.

__Example__ For $\vec{u} = \langle 4, 0, 3 \rangle$, $\vec{v} = \langle -2, 1, 5 \rangle$,

$$2\vec{u} + 5\vec{v} = \langle 2\times4 + 5\times(-2),\ 2\times0 + 5\times1,\ 2\times3 + 5\times5 \rangle$$

$$= \langle -2, 5, 31 \rangle.$$

- The length of the vector $\vec{v} = \langle a_1, a_2 \rangle$ is

$$|\vec{v}| = |\langle a_1, a_2 \rangle| = \sqrt{a_1^2 + a_2^2}.$$

This is the distance between $(0,0)$ and (a_1, a_2), by Pythagorean rule.

Similarly, in 3D, the length of the vector $\vec{w} = \langle b_1, b_2, b_3 \rangle$ is

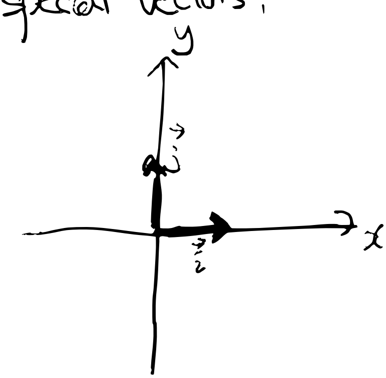
$$|\vec{w}| = |\langle b_1, b_2, b_3 \rangle| = \sqrt{b_1^2 + b_2^2 + b_3^2}.$$

__Example__ The length of $\langle 4, -3 \rangle$ is

$$|\langle 4, -3 \rangle| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5.$$

- Using components, we can name some special vectors, called standard basis vectors.

In 2D: $\boxed{\begin{array}{ll} \vec{i} = \langle 1, 0 \rangle & \text{(positive } x\text{-direction)} \\ \vec{j} = \langle 0, 1 \rangle & \text{(positive } y\text{-direction)} \end{array}}$

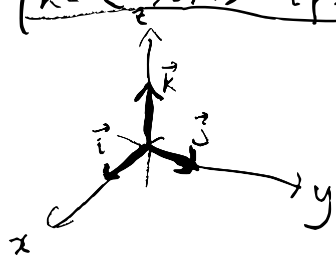


In 3D

$\vec{i} = \langle 1, 0, 0 \rangle$ (positive x-direction)

$\vec{j} = \langle 0, 1, 0 \rangle$ (positive y-direction)

$\vec{k} = \langle 0, 0, 1 \rangle$ (positive z-direction)



By simple arithmetic,

$$\langle a_1, a_2 \rangle = \langle a_1, 0 \rangle + \langle 0, a_2 \rangle$$
$$= a_1 \vec{i} + a_2 \vec{j}$$

$$\langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$
$$= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}.$$

These standard basis vectors are examples of unit vectors, which are vectors of length 1.

Examples of unit vectors

- $\vec{i}, \vec{j}, \vec{k}$
- $-\vec{j}$
- $\langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$
- $\langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \rangle = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{k}$
- ...